Minimal quasivarieties of semilattices with a group of automorphisms

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1996: F-semilattices

Definition

F-semilattice is an algebra $\mathbf{S} = \langle S; \wedge, F\rangle$ where

- $\langle S; \wedge\rangle$ is a semilattice,
- $\mathbf{F} = \langle F; \cdot, ^{-1}, \mathrm{id}\rangle$ is a group,
- **F** acts on $\langle S; \wedge\rangle$ as automorphisms.

For a fixed group **F** the class of **F**-semilattices is a variety.

- Kearnes, Szendrei (97): Self-rectangulating varieties of type **5**.
- Kearnes (1995): Semilattice modes.
- Burris, Valeriote (1983): Expanding varieties by monoids of endomorphisms.
- Ježek (1991): Subdirectly irreducible $\mathbb{Z}$-semilattices.
- Ježek (1982): Simple $\mathbb{Z}^2$-semilattices.

1996: F-SEMILATTICES

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1996: Canonical embedding

Definition

$\mathbf{P}(F) = \langle P(F); \cap, F\rangle$ where $f(A) = A \cdot f^{-1}$ for all $A \subseteq F$.

For $s \in S$ the map $\varphi_s : \mathbf{S} \to \mathbf{P}(F)$, $\varphi_s(x) = \{\, f \in F \mid f(x) \geq s \,\}$ is a homomorphism that separate the points of $\mathbf{S}$.

Lemma

Every subdirectly irreducible **F***-semilattice* **S** *is isomorphic to a subalgebra* $\mathbf{U} \leq \mathbf{P}(F)$ *where*

- $M = \bigcap\{\, A \in U \mid \mathrm{id} \in A \,\} \in U$,
- M *is a submonoid of* F,
- $M \cdot A = A$ *for all* $A \in U$.

If **S** *is finite, then* M *is a subgroup and* $\mathbf{U} = \{\emptyset\} \cup \{\, Mf \mid f \in F \,\}$ *is a flat semilattice.*

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1996: Simple **F**-semilattices

We assume that **F** is commutative (open for general groups).

Lemma (Maróti)

*If a simple **F**-semilattice has a least element, then it is isomorphic to*

$$\mathbf{S}_M = \{\emptyset\} \cup \{\, Mf \mid f \in F \,\}$$

for some subgroup $M \leq F$.

Lemma (Maróti)

*If a simple **F**-semilattice does not have a least element, then it can be embedded into*

$$\mathbf{R}_\beta = \langle \mathbb{R}; \min, F \rangle$$

where $\beta : \mathbf{F} \to \langle \mathbb{R}; + \rangle$ is a homomorphism and $f(a) = a - \beta(f)$.

1996: Simple **F**-semilattices

Definition

$\beta : \mathbf{F} \to \langle \mathbb{R}; + \rangle$ is **dense** if

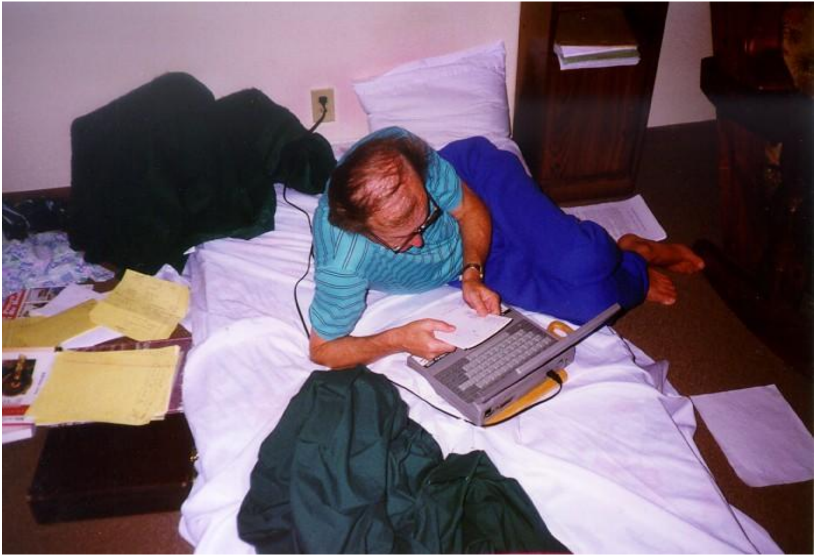
$$(\forall \varepsilon > 0)(\exists f \in F)(0 < \beta(f) < \varepsilon.)$$

Theorem

If **F** *is commutative, then the simple* **F***-semilattices are precisely*

- $\mathbf{S}_M$ *where* M *is any subgroup of* F,
- $\mathbf{Z}_\alpha$, *where* $\alpha : \mathbf{F} \to \langle \mathbb{Z}; + \rangle$ *is a surjective homomorphism,*
- $\mathbf{R}_\beta$, *where* $\beta : \mathbf{F} \to \langle \mathbb{R}; + \rangle$ *is a dense homomorphism.*

These algebras are pairwise nonisomorphic.



1997-2002: Tournaments

Definition

A **tournament** is a conservative commutative groupoid.

1. Ježek, Marković, Maróti, McKenzie (1999):
   The variety generated by tournaments.
2. Ježek, Marković, Maróti, McKenzie (2000):
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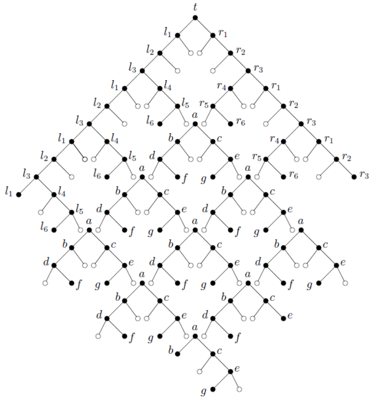
2001: Entropic groupoids

Definition

Medial identity: $(xy)(zu) = (xz)(yu)$, **entropic** identity: you can exchange variables at the same (l, r) position.

Theorem (Ježek, Maróti)

- Decidable of a finite groupoid whether it satisfies all entropic identities.
- Undecidable of a finite partial groupoid whether it satisfies all entropic identities.




PICTIONARY

2007: $\mathbb{Z}$-semilattices

$\mathbf{F} = \mathbb{Z}$, so $\mathbf{F} = \mathrm{Sg}(\{f\})$ for some $f \in F$.

Definition

- $\mathbf{A}_k = \{\emptyset\} \cup \{0, \ldots, k-1\}$ flat semilattice, $f(\emptyset) = \emptyset$ and $f(i) = i + 1 \mod k$
- $\mathbf{A}_\infty = \{\emptyset\} \cup \mathbb{Z}$
- $\mathbf{B}_1^+ = \langle \mathbb{Z}, \min, f \rangle$, $f(i) = i + 1$
- $\mathbf{B}_1^- = \langle \mathbb{Z}, \max, f \rangle$, $f(i) = i + 1$
- $\mathbf{C}_1 = \mathbf{B}_1^+ \times \mathbf{B}_1^-$
- $\mathbf{B}_k^+, \mathbf{B}_k^-, \mathbf{C}_k$ spiral construction:

$$\mathbf{B}_k^+ = \{\emptyset\} \cup \mathbf{B}_1 \times \{0, \ldots, k-1\}$$

$$f(\langle x, i \rangle) = \begin{cases} \langle x, i+1 \rangle & \text{if } i < k-1, \\ \langle x+1, 0 \rangle & \text{if } i = k-1. \end{cases}$$

2007: Minimal quasivarieties

Theorem (Dziobiak, Ježek, Maróti)

The minimal quasivarieties of $\mathbb{Z}$-semilattices are precisely the quasivarieties generated by $\mathbf{A}_\infty, \mathbf{A}_k, \mathbf{B}_k^+, \mathbf{B}_k^-, \mathbf{C}_k$ for all $k \geq 1$. These quasivarieties are pairwise distinct.

Lemma

The unary terms are of the form $t(x) = \bigwedge_{h \in H} h(x)$ for some $H \subseteq \mathbb{Z}$, so they are endomorphisms.

Lemma

Let $\mathcal{Q}$ be a minimal quasivariety, and $\mathbf{A} \in \mathcal{Q}$ be a nontrivial algebra generated by $a \in A$.

- *If $0 \in A$ and $t(a) = 0$, then $\mathcal{Q} \models t(x) \approx 0$.*
- *If $t(a) = s(a)$, then $\mathcal{Q} \models t(x) \approx s(x)$.*

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2007: Minimal varieties

Lemma

Let $\mathcal{Q}$ be a minimal quasivariety of $\mathbb{Z}$-semilattices and $\mathbf{S}$ be the one-generated free algebra. If $|\mathbf{S}| = 1$, then $\mathcal{Q} = \mathcal{Q}(\mathbf{A}_1)$. If $|\mathbf{S}| > 1$, then $\mathbf{S}$ is isomorphic to $\mathbf{A}_\infty$, $\mathbf{A}_k (k \geq 2)$, $\mathbf{B}_k^+$, $\mathbf{B}_k^-$ or $\mathbf{C}_k$.

Theorem (Dziobiak, Ježek, Maróti)

The minimal varieties of $\mathbb{Z}$-semilattices are precisely the quasivarieties generated by $\mathbf{A}_\infty$ and $\mathbf{A}_k (k \geq 1)$.

Remark

There are $2^{\aleph_0}$ many subvarieties of the variety of $\mathbb{Z}$-semilattices.

2009: Minimal quasivarieties of F-semilattices

Again, we have to assume that **F** is commutative.

Lemma

Suppose, that $\mathbf{A}$ *is generated by* $a \in A$, $\mathcal{Q} = \mathcal{Q}(\mathbf{A})$, *and* $\mathbf{B} \in \mathcal{Q}$ *is generated by* $b \in B$. *Then*

$$\varphi : \mathbf{A} \to \mathbf{B}, \qquad r(a) \mapsto r(b)$$

is a surjective homomorphism.

Theorem (I. Nagy)

Suppose that $\mathbf{A}$ *is one-generated and* $\mathcal{Q} = \mathcal{Q}(\mathbf{A})$. *Then* $\mathcal{Q}$ *is minimal if and only if every subalgebra of* $\mathbf{A}$ *generated by a non-zero element is isomorphic to* $\mathbf{A}$.

2009: Minimal quasivarieties of F-semilattices

Theorem (I. Nagy)

If **F** *is finite, then the minimal quasivarieties of* **F***-semilattices are the quasivarieties generated by* $\mathbf{A}_H$ *where* H *is a subgroup of* F*.*

Theorem (I. Nagy)

It is enough to describe all minimal quasivarieties that have no zero element. (Removal of the spiral, construction in both direction.)

Example

$\mathbf{D}_\alpha = \langle \{\langle k + l\alpha, m + n\alpha\rangle \in \mathbb{R}^2 \mid k + l\alpha \leq m + n\alpha\}; \langle min, max\rangle, \mathbb{Z}^2\rangle$ generates a minimal quasivariety for all irrational number α.

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2007: Bi-tournaments

Definition

Every tournament $\langle T; \wedge \rangle$ can be turned into a bi-tournament as

$$x \wedge y = x \iff x \vee y = y.$$

Open problem

Is the variety generated by bi-tournaments is finitely axiomatizable?

Candidate of 12 equations, one of which is

$$\begin{aligned} &g(f(g(x,y), f(f(f(x,y),z), g(x,y))), f(f(x,z), g(x,y))) \\ &= g(f(x, f(f(f(x,y),z), g(x,y))), f(f(x,z), g(x,y))). \end{aligned}$$

Thank you!